The Logic of Quantum Theory Revisited

La lógica de la teoría cuántica revisitada

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Abstract

This article is an unaltered and faithful reproduction of my 1992 MSc dissertation at the London School of Economics (LSE), with very minor editorial amendments, and a new introduction explaining why its publication is timely in the wake of the current debate around the so-called 'adoption problem' in contemporary philosophy of logic.

Keywords: Quantum logic, Philosophy of logic, Anti-exceptionalism, Adoption problem, Hilary Putnam.

Resumen

El presente artículo es una reproducción fidedigna e inalterada de mi disertación, o tesis fin de máster, presentada en 1992 en la London School of Economics (LSE), con mínimas correcciones editoriales. Viene precedida de una nueva introducción que explica las razones que hacen relevante su publicación, derivadas del debate actual en torno al llamado 'problema de la adopción' en filosofía de la lógica contemporánea.

Palabras clave: Lógica cuántica, Filosofía de la lógica, Anti-excepcionalismo, Problema de la adopción, Hilary Putnam.

Introduction to the 2025 publication

It did not occur to me that my old (1992) MSc. dissertation may be worth publishing until in 2024 I came across the collection of papers in *Mind* on the so-called *adoption problem* (Birman, Boghossian and Wright, Devitt and Rose Roberts, Kripke, 2024). Three of those papers, including Kripke's, raise issues that are cognate to distinctions that I drew in 1992, particularly what I refer to as

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the three-layered epistemological model. It struck me when I looked over these papers that the debate regarding whether logic is empirical (and if not, exactly why not) was about to take a new turn, and this time roughly along the lines that I had discussed in 1992. Still, that was not sufficient reason to publish a document with at best historical interest. Further conversations with Alejandro F. Cuesta in the fall of 2024 persuaded me that it may serve some purpose to make this document public. Alejandro was a stellar undergraduate student of mine in the accident-prone 2020 year. My mentoring was interrupted by the pandemic, but we reconnected in the summer of 2024, and quickly saw a path forward for his own PhD thesis on the philosophy of quantum logics. I am happy that we are working together again, and the publication of this document is above all a testament to Alejandro's hard work, interest, and perseverance. (Thanks also to Victor Aranda and Elia Zardini for convening the meeting at Complutense at which this material was presented and discussed on February 12, 2025; the editors of Arif for their interest; and a journal referee for helpful suggestions.)

I have introduced only two sets of changes in the document, which otherwise is an exact rendition of the material that was presented for the award of the *MSc in Philosophical Foundations of Physics* at the LSE in September 1992. First, there is this new introductory preface for the 2025 publication, which very briefly lays out my reasons for thinking that the material is worth revisiting. And second, I have modified and shortened the original lengthy introduction, which was mainly ornamental and full of juvenile self-belief. I am not shamed by it though – I just feel some tender nostalgia for that youth that went away. I was twenty-three years old when I wrote this dissertation, and unbounded enthusiasm has some advantages in those early stages. But there is no point subjecting the reader to it now. Some of the statements in the original introduction are worth retaining, though, particularly as regards the conception of logic as distinct from physical theory –which seems vindicated in the ongoing current debate. There are also of course the grateful notes of thanks to those around me at the time, who certainly deserve to be remembered and acknowledged. So rather than excising the original introduction, I have reduced it to what is publishable *within decorum*. The juvenile references to Nietzsche and Franco had to go, but otherwise, the bulk of the document is entirely unchanged beyond merely editorial corrections. I have not introduced any modification even where I now think modifications are very clearly due. Much has been written since 1992 on Putnam's successive arguments for quantum logic that it would be necessary to take account of now. Here I outline a few of the modifications that seem called for in response to some of the literature in the last three decades. Ultimately, however, they do not alter the conclusions that I reached thirty-three years ago, which makes the publication of this document seem reasonably apt. The material is thus presented here as a historical source, and I am happy that Alejandro F. Cuesta is taking it further in his own PhD work, as outlined in his response, which follows it.

The main claim in the dissertation is that logic is not revisable in the way Putnam famously thought –and this is for some fundamental principled reasons that have to do with the tripartite distinction between logic, empirical physical theory, and the world. So, even an improved interpretation of quantum mechanics, or an ulterior development in physical theory beyond quantum mechanics, could not produce the revolution that Putnam envisaged. Quine's anti-exceptionalism about logic (Quine 1951) is thus implicitly questioned, even though I was very careful in the dissertation – or ever since for that matter –, not to discuss Quine's claims in any depth. (Incidentally, the term "anti-exceptionalism" –as the statement that logic too is subject to the tribunal of empirical evidence– seems to originate in Williamson 2007). I have always found Putnam full of rich argument – perhaps too much argument! – while Quine seemed to trade richly mainly in slogans. The thesis that logic is empirical is no exception. Quine may have established the slogan, but it was down to Putnam to put the detailed arguments forward over the years. If we are nowadays inclined to find against Quine's anti-exceptionalism, it is largely thanks to Putnam's excruciating work, in successive attempts over three decades, at a detailed implementation of the slogan.

However, this is not to say that logic cannot be revised at all. But the revision will not come in the wake of empirical considerations from quantum mechanics, or any other empirical theory for that matter; it will answer to more general philosophical considerations instead. At the time of this dissertation those considerations struck me to specifically concern the status of bivalent metaphysical realism. At any rate, it follows that the revolution has if anything been minimized: A lot more than just the distributivity law must change if logic is to change! Nothing as surgical as the clean excision of the distributivity law, while leaving all the metalogic intact – which is essentially what Putnam aimed for – will do. Whatever formal change there is, it must invariably reflect the underlying nature of the propositions that we already use; so, it must have some reflection at the metalevel. So, a new logic cannot be simply 'adopted' in view of the paradoxes of quantum mechanics. At best quantum mechanics may reveal that we were mistaken about the status of classical logic all along – including the principle of bivalence and metaphysical realism along with it. Either the logical revolution will take metaphysical realism away with it, or it will be no revolution at all! Today I may sound more measured, perhaps, and add a few cautions and caveats, but fundamentally I think that these lessons still stand.

By contrast, Putnam's influential papers (Putnam 1968, 1975) proposed a daring analogy with geometry, which was the focus of the central part II of my MSc dissertation and still seems to me to be at the heart of his proposals. The analogy invited the thought that logic is genuinely empirical – i.e. a posteriori and revisable in response to empirical data – and proven to be so by quantum mechanics, just as geometry is empirical and revisable in response to general relativity. Even further, Putnam advanced the daring claim that classical logic ought to be replaced by quantum logic just so metaphysical realism can be retained. This was meant to be in analogy with how the advent of non-Euclidean geometry allowed Einstein to preserve Galileo's relativity principle, and to retain classical mechanical measuring rods and clocks, in the face of recalcitrant electrodynamic phenomena. With my background in astrophysics and electromagnetism, I was bound to be captivated by Putnam's analogy. I remain in awe at its ambition –even though it is nowadays widely rejected for very good reasons and indeed Putnam himself came to reject it too. The attempt to retain bivalent realism by switching the logic spectacularly fails; but the lesson, I suggested, may be not so much to preserve classical logic together with the uncomfortable paradoxes of quantum mechanics, but to give up on bivalent metaphysical realism instead. I still think there is something to this central claim, although obviously we have become so much more aware of the plethora of distinct alternatives available once Putnam's analogy is thrown out. So, again, today I would introduce several severe caveats, yet stick to the central claim, which seems vindicated.

The background to my developing interest in Putnam's analogy is wrapped up in my decision to move from physics to philosophy for my postgraduate work, so I hope I shall be forgiven for indulging a little memory trip here. (I happen to think that the intellectual content of our ideas is influenced by – and therefore can only be fully understood in reference to – our personal trajectories; but this is a claim for another paper). I came across Putnam's daring analogy in the autumn of 1990, when I also came across him in person. Putnam was giving the Gifford lectures that year in St Andrews, where he was staying for a term, and a workshop had been put together in his honour. I travelled to this workshop in a memorable car journey, together with my Edinburgh tutors in philosophy (Larry Briskman) and physics (Hugh Montgomery). Putnam's lectures were published as his *Renewing Philosophy* (Putnam 1992), while the papers presented at the workshop eventually appeared a few years later in a volume edited by the conference organizers with the title *Reading Putnam* (Clark and Hale 1994). This was my first philosophy conference and catalysed my philosophical career. At this workshop I met Peter Clark, Hilary Putnam, Michael Redhead and Crispin Wright, all of whom would go on to have some significant role in my later professional development. I realize as I write this that I have a huge debt towards these four people, since there and then they determined my future life. (Crossing the Forth road bridge from Edinburgh to St Andrews while contemplating the rail bridge further down the Firth seems to also have had some impact – see the cover of my recent book [Suárez 2024]).

At the workshop, Michael Redhead read a paper criticizing Putnam's proposal for quantum logic, and Putnam responded extensively (Redhead 1994; Putnam 1994), essentially conceding that the analogy with geometry did not hold. Quantum logic was not to quantum mechanics as non-Euclidean geometry was to general relativity after all. But the argument is involved, and Putnam did not concede for Redhead's reasons, but for the different and more conclusive reasons canvassed in his published response (Putnam 1994: 269). The whole exchange fascinated me deeply and for a couple of years afterwards I was immersed in the debate regarding whether logic is empirical. More importantly for the course of my life, in his response to Redhead, Putnam made a point repeatedly to bring to bear a question raised elsewhere by Nancy Cartwright (this is also faithfully recorded in the published version – see Putnam 1994: 173-74). This was the first time I ever heard the name of this American philosopher, and I was very curious. My young and impressionable mind (together with some expert encouragement from Larry Briskman, who was an LSE doctorate) then led me to immediately apply to master's courses at LSE and Cambridge. I got accepted to both and eventually made my way to the LSE to work under Nancy Cartwright, who had just arrived there, while keeping a very friendly attachment to Michael Redhead's Cambridge group, where I often attended their regular weekly seminar. The rest is history, and my life was essentially determined at that momentous event in Putnam's honour at St Andrews in 1990. Quantum logic, Putnam's intriguing analogy, and his involved reasons for defending both, all were certainly in the fray throughout these deeply life-shaping decisions.

I go into these memorabilia not only out of nostalgia, but because Putnam's response to Redhead's talk at that event still strikes me as the most insightful thing he ever wrote on the value of his analogy and the status of quantum logic. Certainly, it is the most sober retrospective analysis of the fortunes of his claims on behalf of quantum logic. His later paper on quantum mechanics (Putnam 2005) is very insightful too, and signals a real change of heart with respect to a range of important issues in the interpretation of quantum mechanics, but it does not discuss quantum logic at all. On the contrary, the whole program of the revisability of logic gets dismissed in a brief parenthetical remark in the abstract: "The quantum logical interpretation proposed in Putnam [1968] is not considered in the present paper, however, because Putnam [1994b] concluded that it was unworkable." Thus, the extended response to Michael Redhead that I watched in awe in St Andrews was Putnam's last word on the topic – and a conclusive word as well.

Guido Bacciagaluppi (2009) has provided a useful framework against which to measure Putnam's most considerate post-1990 position. Bacciagaluppi argues that Putnam sets out to defend three different claims of increasing strength on behalf of quantum logic. Roughly, the first one asserts merely the formal fact that the algebra of quantum operators possesses a non-classical logical structure. This is the inevitable outcome of interpreting the lattice operators in terms of logical connectives, as Birkhoff and Von Neumann (1936) famously did. It is a merely formal result that seems above reproach (although the interpretation of the lattice in terms of logical connectives always seemed to me a considerable assumption – which is undoubtedly the reason why I devoted so much effort in my dissertation to merely establishing this 'formal fact'). Still, Bacciagaluppi is right to grant this assumption, since nothing major hangs on it. The analogy with geometry goes way beyond the merely formal statement that there is quantum 'logic' in the lattice. When Bacciagaluppi mentions the analogy with geometry at this point, he no doubt has in mind only pure geometry. By contrast, Putnam clearly intended his analogy to apply to physical geometry, and to go beyond the statement of the formal differences between Euclidean and Non-Euclidean geometries. Therefore, the analogy does not strictly belong with this first formal uncontroversial assumption. At any rate, I deal with the first formal claim in the more technical part I of my dissertation, where I too happily go along with Birkhoff and Von Neumann's thought that quantum mechanics imports a new logic through its lattice structure.

Secondly, there is the much more expansive additional claim that this 'change of logic' is not merely local, but thoroughly 'global' or even universal, so quantum logic can be seen to replace classical logic *tout court*, as the 'true' logic of thought. This is where the real meat of the argument lies, of course, and upon which the recent debate regarding the 'adoption problem' has focused. My sense is that this is also where Putnam's analogy with geometry most deeply belongs. For the claim advanced on behalf of geometry, recall, is not simply that it heralds a change in formal framework, but that it allows us to reconceptualize the whole of physics, including electromagnetic phenomena, with great conceptual ease, and in agreement with the posit of metaphysical realism that characterizes classical physics. The passage to the new geometry introduced a new paradigm, in Kuhn's celebrated phase, as regards our understanding of the *physical geometry of the world*. (So, this is no longer the first claim about pure geometry). The passage to quantum logic is supposed to similarly herald a new paradigm in our understanding of the logic of thought. Whether these paradigm shifts are in fact comparable – whether quantum logic can be said to introduce a new paradigm – is precisely what is at issue. My sense is that Putnam came to implicitly withdraw the analogy altogether in the response to Redhead in 1994, at the point where he writes: "A realist who rejected the admissibility criterion would have to maintain that although every quantum-mechanical proposition does have a determinate truth-value, it is impossible for us to even guess what those truth-values might be. But this would be to admit that we cannot describe what goes on in a 'quantum-logical world' in the way in which we can describe what goes on in a non-Euclidean world" (Putnam 1994: 279). The admissibility criterion simply stipulates that all elementary propositions are mapped onto the Boolean algebra of bivalent truth values (0 or 1). So, Putnam is implicitly saying that a realist who gave up the criterion would be left with no resources in the meta-language to assert that the replacement of classical by quantum logic is effectively apt for the description of reality (including quantum phenomena). This is strongly reminiscent of the standard formulations of the *adoption problem* (about which more below), and cuts to the heart of the claims in part II of the dissertation.

Finally, the third claim that Putnam made on behalf of quantum logic is that the introduction of the new logic *ipso facto* resolves "the standard paradoxes of quantum mechanics, such as the measurement problem or Schrödinger's cat" (Bacciagaluppi 2009: 2). This one was always the most controversial claim of the lot and has been rightly dismissed by most philosophers of physics. (For informed reviews, see Maudlin 2005, 2022). Putnam clearly and explicitly came to reject it in 1994 but seems to have been doubtful ever since that response to Nancy Cartwright over a decade earlier (Putnam 1981b). It is intriguing to see that this claim did not seem to bother me that much in 1992, and only marginally appears in the dissertation. I certainly was aware that quantum logic could at best block the derivation of the wrong results in, say, the classical description of the two-slit experiment but could not reproduce the quantum predictions. The issue was central to Redhead's talk at the St Andrews workshop (Redhead 1994) and comes up in publications of mine a few years later (Suárez 1996). It is unclear to me though whether I was already then in addition aware of the need to introduce explicit axioms for quantum probability functions defined over the Boolean algebras to obtain the actual quantum predictions for any interference phenomena. Certainly, I recall spending a large part of the following academic year (1992-93) reading up on quantum probability, particularly the works of Luigi Accardi and Itamar Pitowsky, before I made the decision in late 1993 to give up on the initial plan to write a PhD thesis on quantum logic. There were more interesting things going on at the LSE at that time (it was the momentous time of the emergence of the mediating models' movement), and I chose to get involved in the LSE project on models in physics and economics instead.

Still, I retained an interest in Lüders' rule throughout; my PhD thesis dealt with the semantics of quantum theory (Suárez 1997), and I went on to supervise a PhD thesis on the topic some years down the road (Guerra 2009). If it was not at the time of writing this MSc dissertation it must have been very soon afterwards that I read through the seminal paper by Friedman and Putnam (1978). This paper does not appear in the bibliography, but it does turn up in my files from that time with a single annotation on it: "Lüders' rule". Friedman and Putnam reconstruct quantum conditional probability out of Lüders' rule (the standard extension of conditional probability to quantum incompatible observables) and argue that quantum logic can make best sense of it. However, they do not there fully address the problem that quantum logic can do nothing other than simply blocking the undesirable classical statistics. This is a critical point that deserves explaining in some detail.

The difficulty is easy enough to reconstruct in a way sufficiently elementary to teach it in introductory classes on the topic. In the two-slit experiment, where particles go through a first screen with two slits in it, and are later detected at a further detection screen, denote by A, B, and X the following events:

A: the particle goes through the lower slit.
B: the particle goes through the upper slit.
X: the particle is detected in the X region of the detecting screen.

Then according to the *fourth Kolmogorov axiom* of classical probability, which defines conditional probability:

1. $P(X/A) = P(X \wedge A) / P(A)$ and $P(X/B) = P(X \wedge B) / P(B)$.

It also follows from the same axiom that, for the disjunction of A and B:

2. $P(X/A \vee B) = P\big(X \wedge (A \vee B)\big) / P(A \vee B)$.

Now, in agreement with the *distributivity law* in classical logic:

3. $P(X/A \vee B) = P\big((X \wedge A) \vee (X \wedge B)\big) / P(A \vee B)$.

Let us now suppose that the conjoint events $(X \wedge A)$ and $(X \wedge B)$ are *mutually exclusive and jointly exhaustive*, as seems obvious given the set up. Then their probabilities must obey additivity, and we may apply the *third Kolmogorov axiom* of classical probability to find out that:

4. $P(X/A \vee B) = \big(P(X \wedge A) + P(X \wedge B)\big) / P(A \vee B)$.

Now suppose in addition that the single events A and B are also mutually exclusive and jointly exhaustive, and that moreover the source of particles emits them in any given direction at random with the same probability, and that the source is equidistant from both slits, then it follows that:

5. $P(A \vee B) = P(A) + P(B) = 2P(A) = 2P(B)$.

Hence it follows from 4) that:

6. $P(X/A \vee B) = P(X \wedge A) / 2P(A) + P(X \wedge B) / 2P(B)$.

And this expression can now be rewritten as follows:

7. $P(X/A \vee B) = \frac{1}{2} P(X/A) + \frac{1}{2} P(X/B)$.

This last expression (7) is equivalent to the classical mechanical expression for the statistics of classical particles going through one or another slit and then landing on a given region of the detecting screen: $N_{ab} = ½\, N_a + ½\, N_b$. However, the classical statistics are precisely refuted by the experimental evidence, and are of course violated in quantum mechanics, which instead predicts the notorious interference fringes that fail to obey simple additivity. Thus, the above derivation is a *reductio ad absurdum*, and the question becomes which premise, or set of premises, must go. Quantum logicians would blame the step that takes from (2) to (3) since it involves the distributivity law that we know to be false in quantum logic. Putnam follows suit: He claims that the distributivity law is generally false in quantum logic, so its application in the step that takes from (2) to (3) is illegitimate. By contrast, quantum probabilists like Luigi Accardi (1990, 1999) put the blame on the fourth Kolmogorov axiom. This is also known as the *ratio* *definition* of conditional probability and is independently known to be problematic (Hajek 2003). The derivation of the classical statistics is blocked at a different step in the proof, namely the step that takes us from (1) to (2). This allows quantum probabilists to leave the whole underlying Boolean algebra intact, and thus to preserve classical logic, while nonetheless blocking the undesirable derivation of the classical statistics.

Strictly speaking, quantum probability blocks the derivation at every step, since every step involves conditional probabilities. However, in the case of commuting observables, Lüders' rule boils down to the ratio definition (as, more generally, quantum mechanics boils down to classical mechanics in the regime of commuting observables). So, some of the steps are more secure than others. For instance, the very first step that allows us to write down (1) is secure, since X commutes separately with each of A and B. By contrast, the step that takes us from (1) to (2) is problematic since A and B do not commute. But the advantage of quantum probability over quantum logic is that the former gives a recipe for the calculation of the correct experimental statistics, which the latter does not. Putnam can block the derivation of the classical statistics by denying distributivity, but he cannot provide the right probabilities for the two-slit experiment. This simple reason seemed enough to prefer quantum probability over quantum logic. For a long time after I submitted my MSc dissertation, I regarded a revision in the probability calculus as the best way to go, and a much superior response to the paradoxes of quantum mechanics.

Putnam was of course aware of Lüders' rule. In the joint paper with Michael Friedman, they emphasized the difference between quantum probability and classical probability precisely when it comes to its description of *transition probabilities* between eigenstates of non-commuting operators. Lüders' rule is in fact the expression that opens this paper: $Prob_{\varphi}\ (F/E) = Prob_{\varphi_E}\ (F) = \langle\varphi_E | F\varphi_E\rangle$, where φ_E is the normalized projection of φ onto the projection operator E, and "this non-standard conditional probability reduces to the standard one when E and F are compatible" (Friedman and Putnam, 1978: 306). However, the interpretation that Putnam gives to this "transition" probability is certainly not standard. He understands it as a conditionalization of *epistemic knowledge*. In other words, he gives it a subjective interpretation. Thus, Friedman and Putnam write: "All we can do is adjust our probability function to the new information, i.e. we change from $Prob_{\varphi}\ (F)$ to $Prob_{\varphi_E}\ (F)$" (Id. 313). And in revisiting this paper in the response to Redhead in 1994, Putnam is extraordinarily clear and honest: "Referring to $Prob_{\varphi}\ (F/E)$ as a 'transition probability', as I just did, and as Redhead does, may give the impression that it is to be thought of as the probability of a physical change, but Friedman and I thought of it rather as an epistemic probability; the probability that one would assign to F if one starts with the knowledge that the state is φ and one obtains (whether through physical measurement, clairvoyance, a lucky guess, or whatever) the additional information that E." (Putnam 1994: 276).

Indeed, Friedman and Putnam heroically attempt to extract Lüders' rule from the underlying lattice structure of quantum mechanics, to thus be able to claim that *it is the logic that does all the work* in resolving the paradoxes. Yet, quantum probabilists do nothing of the sort. They do not derive Lüders' rule from the underlying lattice structure but rather postulate it as an alternative axiom in a non-classical (or at any rate a non-Kolmogorovian) calculus of probability. These new quantum probability functions then *do all the work*, and their putative representation of genuine transition probabilities (i.e. objective probabilities for genuinely physical state transitions) is at the heart of how they do this work. The underlying lattice in fact can be Boolean and classical, if the probability functions are defined through the new quantum axioms carefully (see Accardi 1990: 122ff.). It is this sort of assumption regarding the nature of probability in quantum mechanics that comes under empirical pressure, not any assumption regarding the underlying logic or our reasoning. Formal probability must, after all, serve as a general template for models of empirical data (Humphreys 2019; Suárez 2020). And if probability theory is understood as a general theoretical framework, then it certainly can come under such pressure. Nothing follows for the underlying logic, never mind the rules of inference that make up our ordinary practices of reasoning.

All of this became very clear to me during the 1992-93 year and determined my decision to abandon the project of writing a PhD thesis on quantum logic. Putnam seems to have been misled by a radical form of philosophical empiricism which, combined with the heroic attempt to retain metaphysical (bivalent) realism, prevents him from seeing how quantum probability can emancipate itself from the underlying logic. True, quantum mechanics imports fundamental conceptual change, but this change does not concern our logical reasoning, or our inferential practices. It is not a change in logic, but it impacts our concept of probability instead (Stairs 1982; see also Guerra 2009 for a detailed study of the underlying philosophical issues regarding this deep conceptual change in probability). And nothing prevents us from using our old classical logic to reason our way into a different set of contingently true axioms for physical probability. Probability certainly can be an empirical theory, even if logic is not.

This is all strikingly in agreement with the recent literature on the *adoption problem* (see the papers in *Mind*, 2024 in the references below) and may be thought to just recapture Kripke's anti-anti-exceptionalist claims in the terms of the philosophy of physics. Hence, there are no real news for the informed reader here. However, it is worth remembering that this consensus had not arrived yet in the early 1990s, and the technical advances in quantum logic were then a miracle to behold. I also did not have the benefit of Kripke's argument which may have had some legendary status, but was not published, so it was hard to know what to make of it. I don't recall Kripke being discussed at all in that exchange on quantum logic in St Andrews in 1990. The name does not appear in the proceedings (Clark and Hale, 1994) except in Blackburn's entry in relation to the private language argument. Nor do I recall spending any time thinking about Kripke's work in the years between 1990 and 1994, other than perhaps for a brief conversation with David Bloor about Kripke's version of the private language argument (Kripke 1982), which Bloor was very keen on – and probably working on at the time: Bloor's book on the private language argument (Bloor 2007) came out soon after. The first rendition of Kripke's argument against Putnam that I ever read was Stairs' (Stairs 2006) and by then I was rightly or wrongly quite disinterested in the quantum logic program, for the reasons I explained above.

Nevertheless, the claim that logic is separate from the empirical world, and our physical theories mediate whatever relation it may hold to the world (i.e. the three-layered epistemological model defended in my dissertation) has the air of an *adoption problem*. But my concern was not so much with a circularity in the sort of reasoning that can conceivably lead us to abandon classical logic. I take this circularity to be very much at the heart of the concerns expressed by both Kripke, on the one hand, and Boghossian and Wright, in a different format, on the other. It certainly is the focus of Lewis Carroll's oft-mentioned paper in this context (Carroll 1895). By contrast, my concern at the time was rather more aligned with Van Fraassen's (1975) point that the 'elementary propositions' of quantum mechanics can only be defined in the metalogic in accordance with some predetermined semantical rules. Birkhoff and Von Neumann expressly chose a bivalent semantics to define 'elementary propositions' and to keep the logical language closed under the usual connectives. But different choices could have been made and can still be made. It is evident that any reasoning that may possibly persuade us to adopt one set of rules over another will have to be carried out in our ordinary logic, i.e. the logic that we use in ordinary inference and reasoning, whatever that logic may be. The fact that I was overly impressed by

Dummett's specific arguments for intuitionistic logic (Dummett 1976) now seems to me incidental to this fundamental point.

Logic is not empirical in the sense that Putnam envisaged it to be, for it can only be dictated by the abstract structure of our empirical theories under some semantic interpretation of the elementary propositions of the theory. And our interpretations as well as our theories are in turn inevitably vastly underdetermined by the actual world. Nor is there a way to argue for the wholesale *adoption* of quantum logic from the practical resolution of the quantum paradoxes. This requires a quantum probability calculus that can perfectly well remain independent of the underlying logic in the lattice of propositions. The analogy with geometry thus breaks down, since physical geometry belongs in the second layer of the epistemological model, as a constitutive part of our physical theories. Our reasoning patterns and rules of inference are therefore not at stake in a change in the geometrical description of spacetime.

General relativity came to be accepted without anyone being forced to change the way they *reasoned* – in fact it came to be accepted through reasoning from evidence carried out in accordance with our ordinary rules of inference. Yet our rules of inference are precisely what is at stake in any putative change in the underlying logic of 'our thought' – whether this locally concerns our reasoning about quantum mechanical 'elementary propositions' only, or globally all our inferential practices at large. The three-layered epistemological model makes it clear that logic goes way beyond what any physical theory can possibly determine, anyway. It informs our judgements regarding fictional and hypothetical possibilities beyond what is strictly physically possible in any legitimate sense of the word. (A point elegantly carried home also by Boghossian and Wright 2024: 105).

Nowadays, I like to impress the fact that the rules of reasoning making up our inferential practices, however loose or hard to determine, inform our scientific theories and models, including quantum theoretical models. A scientific theory is essentially nothing but a sophisticated and complex tool for inference (Suárez 2024). Now revisiting my old work on quantum logic, I am forced to wonder how much of this may have remote ancestry in Putnam's original views on quantum logic. The part of Putnam's program that still seems unquestionable is the fact that there is a distinct logic, and a distinct probability calculus, 'within' the quantum mechanical formalism. But if this 'logic' is to genuinely inform our reasoning in the domain of quantum phenomena, it ought to be *in addition to*, never *in place of*, our usual inferential practices. That is, I assume that whatever quantum logic or probability may be put to effective use (in our inferences regarding the phenomena that it describes) it must include our usual logical reasoning practices, which always stand in the background, yet are not readily formalizable (Kripke 2024; Padro 2024). There are additional rules of inference (that is, additional to standard classical logic) that are imported by every model, which govern our reasoning to the properties of the target in the model's domain of application (this is known as surrogative inference). These rules of inference must be taken in a sufficiently loose sense, and are based upon exploratory analogies, but are nonetheless normative of their own right (Suárez 2024: Ch. 7 and 8, particularly 217-220). So, there is one sense in which Putnam was right after all: An empirically confirmed scientific theory can certainly inform our inferential practices, even if – contrary to what Putnam thought – it does not thereby impact our most general or basic rules of inference, i.e. logic.

The practices of inference that we call 'logic' stand at the most general end of the spectrum of our inferential practices and are precisely those that cut across any possible scientific theoretical representation of the world. So, they cannot be impugned by the empirical evidence that favours any given model or theory. This point has some implications for the arguments against the distinction between 'accepting a theory' and 'adopting a logic' (Williamson 2023), since it shows that the distinction essentially boils down to an issue of generality or universality of the rules of inference under discussion. In my view one must *accept* quantum theory in Van Fraassen's sense of the term, as the empirically adequate theory of its domain (Van Fraassen 1980). But one need not thereby *adopt* quantum 'logic' other than in the trivial sense of accepting the fact that the lattice structure of the theory imports a sized-down or truncated version of the axioms of logic or an expanded version of the axioms of probability. Logic cannot be empirically revised as Putnam thought, but the rules of surrogative inference that are valid in any given application of a scientific representation of course are so revisable and are often revised. At this point – and only, it seems to me, at this point – the analogy with geometry has bite. (Just think about how you would calculate the shortest path in Euclidean and non-Euclidean geometries. That is where the rules of inference made available by each theory can make a practical difference.)

Putnam started off, as any Reichenbach student would, with the proposal of a three-valued logic specifically for quantum mechanics (Putnam 1957). He diverted at Harvard under the spell of Quine's conventionalism to defend a much more radically empiricist proposal for logic in general (Maudlin 2022). In wanting to fit in with Quine's rejection of the distinction between analytical and synthetic knowledge (a distinction dear to the European logical empiricists), Putnam was driven to the radical view that quantum mechanics requires a change in the logical framework at the core of Quine's web of belief. By the time I got to know him better (Putnam was the sponsor of two of my periods as visiting scholar at Harvard –in 2009 and again in 2011), he had left both quantum logic and metaphysical realism behind. One wonders how much more viable and balanced his views would have stayed throughout had he stuck to the Neo-Kantian roots of his beloved teacher and mentor. There may be a lesson here for us all.

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The Logic of Quantum Theory: A Minimized Revolution

A selective review of the philosophical implications of quantum logic

MSc. Dissertation in the Philosophical Foundations of Physics, London School of Economics, September 1992.

Original 1992 Introduction (downsized for the 2025 publication)

My interest in quantum logic started while I was studying physics at Edinburgh University. Marc Denervaux knew that I was helping to produce a version of an EPR experiment, and he once brought me a pile of papers on quantum logic. I just looked through them and decided that they would not be very useful for the experiment, so I returned them. But my curiosity had been aroused. What struck me first was –there must be something *wrong* with the idea of a quantum logic. At the time, quantum mechanics seemed as the paradigm of extreme empiricism; I did not even think of it *as a theory*, –and, in fact, I still believe few physicists think of quantum mechanics as a theory at all–. Quantum mechanics was just a set of recipes to find one's way in the laboratory. Even when I specialized in Astrophysics, quantum mechanics was only 'that set of useful tricks that allowed one to find the expected spectral lines in the correct places in the photographic plate'. General Relativity had a totally different taste: it was an abstract, high-level theory, a product of the deductive genius. But quantum mechanics… –well, quantum mechanics was only a (powerful) *tool*.

I only began to hint at the importance of alternative logics through reading Wittgenstein (1922); the Tractatus surely constitutes an attempt to study logic from a global perspective. It is remarkable that the last two sets of propositions of the Tractatus (6 and 7, most particularly 6.1. on tautologies, 6.3. on probability, 6.4. on hierarchies in logic and 6.5. on whatever lies outside logic –what Wittgenstein calls *mysticism*.) have been the subject of so little attention from logical positivists given that they effectively define the limits of logic. The point lies, however, in the general "flavour" of the discussions. The picture that comes out from the book is a tantalizing one: –logic is, in fact, unavoidable. Logical inference mediates every thought, whether conventional or not, and it is the basis for a strong theory of meaning: The meaningfulness of a discourse is defined according to the validity of its logical structure.

A deep insight into the implications of quantum logic for the question of the status of logic, is contained in Michael Dummett's splendid paper, "Is Logic Empirical?", first published in 1976 as a response to Hilary Putnam's (1968) claim that quantum logic proves logic to be empirical. This paper is a thrilling and wonderful piece of understated skeptical criticism, and it seems to put forward a position more akin to the Tractatus than to the later Wittgenstein ideas. I think it is fair to say that Dummett's paper has played a central role in the shaping of my dissertation. It has helped me to understand the priorities in my research and what issues, among the huge literature in quantum logic and the philosophy of logic, I ought to emphasize.

When I first conceived the idea of writing on quantum logic, I took it for granted that the result would be a highly technical piece of work. Problems in the foundations of quantum mechanics are intrinsically related to lattice theories and algebras, and, of course, to the formalism of dual evolution by Von Neumann[1]. Lattice theory is the natural foundation for work in quantum logic and algebras constitute the starting point of the so-called algebraic approach to quantum mechanics. The fundamental discovery of Birkhoff and Von Neumann, that quantum mechanics can be written in terms of propositions, and these can be related to the underlying mathematical structures, as expressed in their 1936 paper, "*The Logic of Quantum Mechanics*", is the basic historical landmark. This paper is, still to the present day, one of the most fascinating and clarifying texts on the subject. I therefore dedicate a full section to its exposition, even if acknowledging lack of understanding of a few of its parts.

Of course, reading Dummett's paper put me off the idea of writing a straightforward technical dissertation. For a time, I had considered researching the Kochen-Specker paradox or studying the probability theory in quantum mechanics

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[1] This formalism, as presented by J. von Neumann in his (1932) *Mathematical Foundations on Quantum Mechanics*–, provides quantum mechanics with the necessary requirements to fulfil the conditions to be properly addressed as a theory. Such requirements are of course of a logical nature: they must specify the conditions under which the central equations of the theory apply. Von Neumann, in the above-mentioned work defined the circumstances (measurement) under which the fundamental equations (the Schrödinger equation) fail to apply. Needless to say, I realized how much **theory** there is to quantum mechanics when I learnt the Von Neumann formalism.

(probability can be defined as a measure on the lattice, so the connection with the logic can be immediately established). But the realization that the foundational problems of quantum logic were of a massively semantical nature (and therefore more likely to be comprehended and adequately tackled by purely philosophical tools, such as a theory of meaning) led me to a sort of compromise.

The compromise consists in keeping a strong technical exposition of the logical structure of quantum mechanics, but restricting the emphasis of the discussion to questions related to the implications of quantum logic for the philosophy of logic. The dissertation is thus structured in two parts; the first introduces the fundamental notions of quantum mechanics (section 3) and describes how these can be related to an underlying logic of propositions (section 4). Section 1 contains an outline of some personal thoughts about the role of logic in epistemology, and about the place that quantum logic holds in the context of the philosophy of physics. Lattice theory plays a fundamental part in the construction of quantum logic –it may be regarded as providing the **semantics** for the quantum-logical language–. Therefore, I have not thought it convenient to give lattice theory a marginal treatment by confining it to an appendix, –as is usually the case in most standard expositions. It is thus introduced extensively, as a system on its own, and as quickly as possible, in section 2. A word of warning is due here with respect to notation: the symbol ∈ is used all throughout the dissertation for set-inclusion (**A** ∈ **B** means that **A** is in **B**).

The transition to the more philosophically oriented second part of the dissertation is smoothly carried out by a discussion of Van Fraassen's (1975) excellent paper, "*The Labyrinth of Quantum Logics*" (again in section 4), where the distinction between the syntactic and the semantical aspects of quantum logic is clearly stated, and the significant stress is already anticipated to lie on the latter. Part II of the dissertation contains the properly philosophical discussions. First, the empirical view of logic is put forward, following Putnam's many contributions to the subject (section 5). It is then shown that this view runs into problems when considering arguments from the theory of meaning, according to the general discussion by Dummett, and to the more specific result in Bell & Hallett's 1982 paper, "*Logic, Quantum Logic and Empiricism*" (section 6). Finally, in section 7, in the spirit of reaching some conclusion, I give a very brief historical digression and suggest a somewhat unusual interpretation of the aims of quantum logic.

The last year at LSE has been a happy period in my life. I have learnt methods to approach theoretical research in philosophy, especially research concerned with logic and probability. Moreover, I have learnt that the fundamental questions, –in science in general, and in physics in particular-, are always of a philosophical nature, and I have abandoned my quite naive operationalist views –which in no way implies that I have become a realist. I shared many discussions with Phillip Hübner, who would probably disagree with many ideas in this dissertation. I certainly learnt a lot of Austrian mathematical physics from him. Giovanna Corsi, of the University of Florence, introduced me to lattice theory. My tutor at LSE, Elie Zahar, had an enthusiasm for consistency that proved to be contagious. My supervisor, Nancy Cartwright, saw me through the conceptual jungle of the formalism of quantum mechanics. Of course, none of them are responsible for possible errors.

London, September 1992

Part I: Quantum Logic Revisited

1. Logic in Physical Theory

The following questions might seem a natural preamble to the discussion of quantum logic: What is the role of logic in the theories of physics? How does logic relate to the experimental practice of physics? What could be the epistemological importance of quantum logic? This section aims at a very brief elucidation of some of these questions. The intention is to present a natural philosophical introduction to quantum logic, although it must be stressed that the questions dealt with in this section are, on their own, part of important contributions and debates in modern philosophy of science. The approach taken here is highly personal and, admittedly, biased towards quantum logic.

Logic states the laws of thought. Whether these laws can be interpreted from the point of view of a realistic correspondence theory of truth or not is –partly– the subject of the discussion of part II of this dissertation, and we should not be concerned with that question at this stage. Human beings use logic to make inferences and to construct theories about the world. However, a theory is independent of the logical processes that have been followed in order to produce it[2]. Theories are elaborated in a social context, so they tend to accommodate certain social (*external)* constraints and, also, to account for some empirical results. These external constraints do not need to be of identical logical nature as the laws of thought. If they were of identical logical nature, then, by implication,

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[2] See for example Popper (1975: 106-152).

the social world would be regulated by actions that preserve logical validity and truth; this is not an assumption that can be safely guaranteed. Similarly, neither are the empirical constraints required to share the same logical nature as the laws of thought. If they were so required, then it would have to be accepted that the world is logical in itself: –another clearly unjustified assumption.

Occasionally, theories are created with a view to accommodate some well-known experimental conditions. Thus, such theories already contain, in their original axiomatic structure, some schemata that fulfils certain empirical constraints. If the view is taken that the laws of thought are *not* subject to those external or experimental constraints (or at least not *immediately)*, then the following possibility arises that, in certain cases, the logic of thought might be inconsistent with the logical structure of some theory. This is exactly what has occurred with the quantum theory. The logical basis of the theory (quantum logic) is *not* consistent with the logic of thought (Aristotelian logic).

There is an epistemological framework that accommodates the above interpretation of the philosophical importance of quantum logic. It resembles Popper's three worlds ontology[3], but contains significant differences. I prefer to call it the 3-layers epistemological model, because it does not make, –as opposed to Popper's scheme–, any ontological claims: the *layers* in this model do not have to be interpreted in such strong terms as *worlds*. The model is thus a purely epistemological one, and I accept it can be biased: I use it to explicate how quantum logic arises.

The first layer contains *thinking*. This is defined as the production of concepts and the establishment of correct inferences between premises and conclusions in accordance with the laws of (Aristotelian) logic. Thus, every thought has a structure, even prior to its being put into words[4]. The structure is always of a logical nature, –that logic being, of course, the *classical* (Aristotelian) logic.

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[3] See footnote no. 2.

[4] N.B. the fundamental distinction between the earlier and later Wittgenstein. The later Wittgenstein has often been interpreted as claiming that the structure that provides meaning belongs to the realm of language, not of thought –there is no *logical* structure, only *grammatical* rules. Thinking is thus independent of logic; it is by "wording" the thought, –the positioning into a grammatical structure–, that we convey meaning. Bartley, for example, interprets the later Wittgenstein as having meant that "Even the basic laws of logic […] were now to be regarded as conventions, as highly systematic schemes for ordering statements which were […] in no way more basic than language-games." (Bartley 1973: 160). If Bartley's interpretation is correct, then the claim that the *Philosophical Investigations* is a natural continuation of the *Tractatus* would not seem to hold water. In this section I am obviously following the Tractatus.

The second layer contains *theories*. In the 3-layer epistemic model a theory always has a logical structure, although this structure need not agree with classical logic. The assertion that a theory always has a logical structure follows from Ramsey's (1929) paper "*Theories*". This assertion is at the core of contemporary work in the structuralist school in the philosophy of science[5]. The logical structure of a theory can be easily derived from the mathematical axioms upon which it is founded. Different sets of axioms can still represent the same logical structure, so the logical structure of the theory is uncovered only when a *morphology*, –not just an *axiomatization* of the theory–, is defined. The morphology can be explicated in terms of set-theory and the assumption that set-theory implies a logical calculus through relations of set-inclusion, enables one to perceive the morphology as the representation of the logical structure of the theory.

The third layer is the factual empirical collection of data. The definition of the contents of this layer is certainly a matter of discussion: some authors would claim that in fact, there is no such factual empirical collection of data. According to this view, all data is theory-laden[6]. However, an alternative view is contained in Ramsey's (1929) paper and has consequently been maintained by many logical positivists; this view holds that at a very low level in the empirical scale, there are so-called *observables*. I have supported this view elsewhere while stressing that what scientists normally call *observables* are not observables at all in the Ramseyan sense.[7]

Nevertheless, this discussion does not affect the explication of quantum logic, in the sense that quantum logic emerges from a relation between the first two layers and is in no way influenced by the third layer. This is, in fact the main achievement of the 3-layer epistemological model. It distinguishes strongly between the questions dealt with by the logical structuralists and quantum logicians, and those questions addressed by empiricist philosophers of science. The former discuss issues such as the logical consistency of thought and theory. The latter discuss issues such as realism and empirical adequacy between theory and facts. This is relevant to my

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[5] See, for example the book by Balzer, Moulines and Sneed (1987). I learnt about logical structuralism from (Moulines 1992).

[6] Of course, Popper, Kuhn, etc.

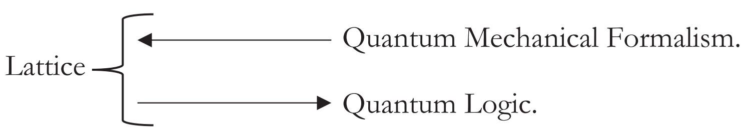
[7] In my seminars in the philosophy of cosmology at LSE, I made the distinction between Primary Observables, –observables in the sense of Ramsey– (such as measurements of frequencies, angles, intensity of light, polarization, etc.), and Secondary Observables (such as the Hubble cosmological constant and the density of the universe parameter). Scientists normally also call the latter "observables", but they are derived from the Primary Observables by very low-level theoretical manipulations such as Doppler shifts and equivalent widths.

dissertation in the sense that it shows that there exist two definitions of realism. One can define realism by establishing the empirical predictive power of a theory –i.e. by observing the relationship between the second and the third layers of the epistemological model. But one can also define realism by characterizing a logical correspondence between statements that result from the theory and the principle of bivalence (i.e. by assuming statements are necessarily either true or false), –this only concerns the model's first and second layer. This distinction between the meanings of realism is often not sufficiently emphasized. However, it is a crucial distinction for the purposes of this dissertation. I will return to it in Part II, when I will discuss the philosophical implications of quantum logic for logic and realism. The rest of Part I is dedicated to a technical exposition of quantum logic.

A final remark concerns the relationship between logic and the world. It is important to stress that in the 3-layer epistemological model there is no direct relationship to be established between layers 1 and 3: The empirical data (presumably extracted from the world) is in no way dependent upon the laws of thought. But the opposite holds as well: the laws of thought are independent of the empirical data, that is, logic is independent of the world.

2. Lattice Theory for Quantum Mechanics

Lattice theory is the cornerstone of quantum logic. A propositional logic can always be read off a lattice. This is the case because a lattice defines set-theoretic inclusion relationships in a partially ordered set and a logic can always be derived from set-theoretical inclusion relationships. It so happens that the formalism of quantum mechanics can be expressed in algebraic terms, so a lattice can be given for quantum mechanics. This closes the circle: the formalism of quantum mechanics can be reduced to a lattice which, in turn, implies a logic. The lattice is therefore the fundamental structure: it gives rise to quantum logic:



In this section I will present the fundamentals of lattice theory, by way of stating well-accepted definitions[8]. In the following two sections I will describe the formalism of quantum mechanics and quantum logic.

[8] A good summary of lattice theory is in Jammer (1974: 523-527). Michael Redhead's (1987: 176) has a very brief but clear appendix on lattice theory.

Definition 1: Partial Ordered Set (Poset).

A poset is a set A on which a relation $\leq$ has been defined, such that $\leq$ is reflexive, transitive and antisymmetric.

1. Reflexive: If $a \in A$, then $a \leq a$.
2. Transitive: If $a, b, c \in A$ then: if $a \leq b$ and $b \leq c$ then $a \leq c$
3. Antisymmetric: If $a, b \in A$ then: if $a \leq b$ and $b \leq a$ then $a = b$

Definition 2: Meet and Join.

A meet is also called the "greatest lower bound". The meet **x** of two elements of A, say a and b, is defined as follows:

$\mathbf{x} = \text{meet}(a, b)$ iff $\quad \mathbf{x} \leq a \;\&\; \mathbf{x} \leq b \quad \&\; \forall c \in A\ (c \leq a \;\&\; c \leq b \rightarrow c \leq \mathbf{x})$

A join is also called "the lowest greater bound". The join **y** of two elements of A, say a and b, is defined as follows:

$\mathbf{y} = \text{join}(a, b)$ iff $\quad \mathbf{x} \geq a \;\&\; \mathbf{x} \geq b \quad \&\; \forall c \in A\ (c \geq a \;\&\; c \geq b \rightarrow c \geq \mathbf{y})$

The following notation is conventional:

$\mathbf{x} = \text{meet}(a, b) = \mathbf{a \;\&\; b}$
$\mathbf{y} = \text{join}(a, b) = \mathbf{a \vee b}$

This notation already incorporates the fact that a logic can be abstracted from a lattice. The meaning of the connectives **&** and **v** is thus defined from set-theoretic inclusion relationships.

Definition 3: Lattice.

A lattice Æ is a poset such that all meets and joins belong to the poset. It is defined as follows:

Æ = $\langle A, \leq \rangle$ is a lattice iff:

1. $\langle A, \leq \rangle$ is a poset.
2. $\forall a, b \in A\ (\mathbf{x} \in A \;\&\; \mathbf{y} \in A)$.

Or in the conventional notation:

2. $\forall a, b \in A\ \big((a \;\&\; b) \in A, (a \vee b) \in A\big)$.

Definition 4: Maximun and Minimun of a Lattice.

A lattice $Æ = <A, \leq>$ has maximun and minimun iff:

1. There is an element in A, say **1**, such that all elements in the lattice are "below" it.
   So, **1** is a maximun iff **1** ∈ A & ∀ a ∈ A $(a < 1)$.
2. There is an element in A, say **0**, such that all elements in the lattice are "above" it.
   So, **0** is a minimun iff **0** ∈ A & ∀ a ∈ A $(a > 0)$.

Definition 5: Complemented Lattice.

If $Æ = <A, \leq>$ is a lattice with maximun and minimun, then the complement of any element can be defined as follows:

a' is the complement of a ∈ A iff a & a'= **0** and a v a'= **1**.

Æ is a complemented lattice iff every element in Æ has a complement.

Definition 6: Distributive Lattice.

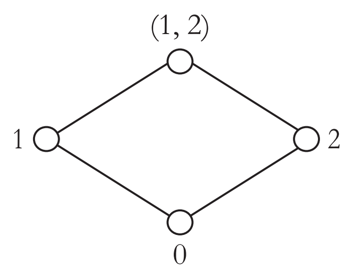
A lattice Æ is distributive iff ∀a, b, c ∈ A the following hold:

1. a & (b v c) = (a & b) v (a & c).
2. a v (b & c) = (a v b) & (a v c).

Definition 7: Boolean Algebra.

A Boolean algebra is a complemented distributive lattice.

Lattices can be drawn schematically by so-called Hasse diagrams. A straightforward example of a Boolean algebra containing two elements {1, 2} can be easily constructed:



Definition 8: Lindenbaum Algebra.

The algebra of classical propositional logic is a Lindenbaum algebra. A classical algebraic model <B, I> consists of the algebra of truth-values B = <{0,1}, v, &, ¬> together with an interpretation I: P → {0,1}, –where P is the set of propositions[9]–, such that the following conditions are fulfilled:

1. I (T) = 1
2. I (F) = 0
3. I (¬ p) = ¬I (p)
4. I (p & p') = I (p) & I (p')
5. I (p v p') = I (p) v I (p')

A Lindenbaum algebra is an algebra **L** of propositions: **L** = <**≈**, +, ×, –, 0, 1>, – where ≈ is an equivalence relation–, such that the following can be defined:

1. 1 = [T]
2. 0 = [F]
3. **¬**[p] = [¬ p]
4. [p] × [p'] = [p & p']
5. [p] + [p'] = [p v p']

Corollary 8b: A Lindenbaum algebra is Boolean.

Definition 9: Modular Lattice.

(b,c) is a modular pair if ∀ a ≤ c (a v (b & c) = (a v b) & c).
A lattice is modular iff every two elements of it are a modular pair, that is iff a ≤ c implies a v (b & c) = (a v b) & c for all a, b, c of the lattice.

Corollary 9b: Every distributive lattice is modular.

However, the opposite does not hold: some modular lattices are not distributive. Modularity is thus a **weaker** requirement than distributivity.

Definition 10: Involutive Dual-Automorphism.
A homomorphism is a mapping h: $L_1 \rightarrow L_2$ of a lattice L_1 into a lattice L_2 such that the following hold for all a, b ∈ L_1:

[9] *Propositions* are defined by their having a definite truth-value associated to them (a proposition always has some truth-value).

1. h (a v b) = h (a) v h (b).
2. h (a & b) = h (a) & h (b).

An isomorphism is a one-to-one homomorphism. An automorphism is an isomorphism of the lattice with itself. A dual-isomorphism is a one-one mapping d: $L_1 \rightarrow L_2$ such that $a \leq b$ implies $d(b) \leq d(a)$ for all a, b $\in L_1$. A dual-automorphism is a dual-isomorphism of a lattice with itself.

An **involutive dual-automorphism** of a lattice L is a dual-automorphism d such that $d(d(a)) = a$ for all a $\in$ L.

Definition 11: Orthocomplemented Lattice.

A lattice L is orthocomplemented iff it possesses an involutive dual-automorphism which satisfies the following:

$a \leq d(a) \rightarrow a = 0$ for all a $\in$ L.

We denote d(a), –or a'–, as the orthocomplement of a. The operation d, –or '–, is an orthocomplementation.

Definition 12: The algebra of Hilbert Space.

The algebra of the subspaces of a Hilbert space is an orthocomplemented modular lattice (orthomodular lattice).

The proof and significance of **definition 12**, as asserted by Birkhoff and Von Neumann, is the subject of discussion in section 4 of this dissertation. First, in section 3, the formalism of quantum mechanics is introduced.

3. The Von Neumann formalism of Quantum Mechanics

Von Neumann introduced the notion of dual evolution in 1932[10]. In the Von Neumann formalism, a quantum mechanical system obeys Schrödinger's equation except when subject to an interaction. There are thus two different rules by which a system, described by a statistical operator **U**, might evolve. The first one transforms the original **U** into some $\mathbf{U_t}$ in a causal, deterministic fashion, as prescribed by Schrödinger's wave-mechanics:

$$\mathbf{U_t} = \exp(-2\pi itH/h)\, \mathbf{U} \exp(2\pi itH/h) \quad (1)$$

[10] The discussion in this section follows (Von Neumann 1932) and (Redhead 1987).

The second one transforms the system from **U** into $\mathbf{U_t}$, in an indeterministic, acausal and instantaneous manner. Moreover, it transforms pure states into mixtures[11].

$$\mathbf{U_t} = \Sigma_{n=1} (U\Phi_n, \Phi_n) \, P \, [\, |\Phi_n>] \qquad (2)$$

A measurement has two parts: an interaction and an observation. The interaction, in the Von Neumann formalism, has the effect previously described: the pure state becomes a mixture. The observation, on the other hand, has the effect described in quantum-mechanics as the *collapse of the wavefunction*: one particular eigenstate is selected. The mathematical means to represent such action (the projection operators) were also given by Von Neumann. In equation (2) above, $P\,[\,|\Phi_n>]$ is a projection operator; specifically, the one that projects any vector in Hilbert Space into the subspace spanned by $|\Phi_n>$.

A projection operator or a *projector*, **P**, is an idempotent Hermitian operator. Idempotency implies that $\mathbf{P = P^2}$. An operator is Hermitian (or self-adjoint) iff it is equal to its adjoint: **P = P**, where the adjoint is defined as follows:

$$<x\, P^{\lrcorner}\,(y)> = <y\, P(x)>^*, \forall x, y \in \mathbf{V}$$

Here: < | > indicates Dirac inner product, * is the complex conjugate and **V** is the Hilbert space on which we are operating.

Several interesting results can be seen to apply to projectors. If the initial wave-function is given by

$\mathbf{|\phi> = \Sigma_{i=1}^{N} c_i |q_i>}$ and the observable we are interested in is given by $\mathbf{Q\,|q_i> = q_i\,|q_i>}$ then we can always write a projector as follows: $\mathbf{P\,[\,|q_i>] = |q_i> < q_i|}$.

[11] Or mixtures into different mixtures. In this dissertation the convention adopted in quantum logic is followed, –unless otherwise stated–, to assume that the quantum mechanical system is in a pure state. Most authors take this convention for granted (e. g. Van Fraassen 1975), but it is by no means clear to me that such convention is not eventually going to affect the discussion. On the contrary, it seems meaningful to argue that, since quantum logic sets conditions for truth and validity of quantum mechanical statements, it might become relevant to discuss whether situations can arise in which no true description can possibly be extracted from a pure state, but such description is represented by a mixture (given that the ignorance interpretation of mixtures is discarded). As regards this possibility, (Prigogine 1984), for example, argues for the physical non-existence of pure states; in his description all quantum mechanical states are mixtures.

This is certainly a projector for, if applied to any wavefunction $|\phi>$, it gives the component of the expansion associated with the relevant subspace:

$$\mathbf{P}\,[\,|q_i>]\;|\phi> = |q_i> < q_i\,|\phi> = c_i\;|q_i>.$$

The domain of the projector is the whole vector space **V**. The range is the one-dimensional subspace spanned by the particular eigenvector. It comes then as no surprise that $\Sigma_i\,\mathbf{P}\,[|q_i>] = \mathbf{I}$ where **I** is the identity operator.

Projectors obey orthogonality relations:

$$\mathbf{P}\,[\,|q_i>]\;\mathbf{x}\;\mathbf{P}\,[\,|q_j>] = \delta_{ij}\,\mathbf{P}\,[\,|q_i>]$$

And their eigenvalues must be 1 or 0:

$$\mathbf{P}\,[\,|q_i>\,]\;|q_i> = q_i\;|q_i> = \mathbf{P}^2\,[\,|q_i>\,] = q_i^2\;|q_i>$$

by the condition of idempotency, so $q_i^2 = q_i$ and that implies $q_i = 1$ or $q_i = 0$.

If two projectors commute: $\mathbf{PQ}\,(|q_i>) = 1\;|q_i>$.
If they don't commute, then: $= 0\;|q_i>$.

The **spectral theorem** is a central result of the Von Neumann formalism. It expresses the peculiarity that, any observable can be written in terms of projectors:

$$\mathbf{Q} = \Sigma_i\;q_i\;\mathbf{P}\,[\,|q_i>].$$

Finally, it is convenient to remark that it is possible to write projectors for the case of degenerate eigenvalues as well. Redhead denotes such cases as follows:

$$\mathbf{P}_Q\,(q_i) = \Sigma_{j\,|\,qj\,=\,qi}\,\mathbf{P}\,[\,|q_i>]$$

where the range of the projector is the subspace spanned by all the eigenvectors $|q_i>$ with $q_j = q_i$ (a subspace is spanned by a set of vectors if every element in the subspace can be expressed as a linear combination of members of this set).

4. The Logical Structure of Quantum Mechanics

This section contains a discussion of the classic 1936 paper by Birkhoff and Von Neumann, "*The Logic of Quantum Mechanics*" (from now on: LQM). LQM historically set the arena for the vast majority of the contributions in the field of quantum logic. More to the point of this dissertation, LQM already contains the main ideas upon which Hilary Putnam and David Finkelstein base their arguments for the empirical character of logic. The arguments by Putnam and Finkelstein will be discussed in section 5, and the claim that the authors derive from these arguments will be shown (in section 6) to have been put under serious pressure (if not just to have been simply refuted) by more general philosophical considerations about the status of logic. Some of the counterarguments used against Putnam and Finkelstein will, in turn, prove to have roots in some of the implicit ideas contained in the original LQM paper by Von Neumann and Birkhoff.

Therefore, this section sets the discussion for the rest of the dissertation. Some of the issues that will be reviewed in posterior sections are already anticipated here –e. g. the important notion that the conceptual changes brought about by quantum logic are of a fundamentally *semantical* nature. This anticipation is achieved via the insertion of comments and explanations from Van Fraassen (1975). In his paper, Van Fraassen manages to prove convincingly that there exist several choices for the construction of a quantum logic. Birkhoff and Von Neumann choose a specific set; others, like Reichenbach, make different choices. Even more significantly, other choices are still available: –at least by 1975, quantum logic was not yet an exhausted discipline.

Quantum logic springs from the possibility of finding a relationship between elementary statements of quantum mechanics (what I shall call the *propositions* of quantum mechanics) and the mathematical abstract space upon which the formalism of quantum mechanics is founded. Such possibility arises equally in classical mechanics, in electromagnetism, or in thermodynamics. The fundamental difference is that the propositional calculus rendered by any classical theory of physics is a Boolean algebra, while the underlying lattice corresponding to quantum mechanics is an orthocomplemented modular lattice (if the Hilbert space is taken to be finite-dimensional).

One of the aims of LQM is to prove our **Definition 12** in section 2, namely that the algebra of subspaces of a (finite) Hilbert space is an orthomodular lattice. Another aim is to show what the lattice of a Hilbert space is **not**, namely it is definitely *not* a distributive lattice, and cannot consequently be Boolean. The importance of this negative result is great: –the logic of quantum mechanics is totally different to the logic of any other classical theory, and, more importantly, it is different the logic expressed in the propositional calculus. I shall not stress this point any more in this section: from the results of lattice theory, it is clear (Definition 8) that the Lindenbaum algebra is Boolean; so, if the lattice of the subspaces of Hilbert space is not Boolean, then it is clear that quantum mechanics does not correspond, in logical terms, to the propositional calculus.

For the time being, I shall just review the proofs contained in LQM of the negative result (*the lattice of quantum mechanics is not Boolean*) and of the positive result (*the algebra of the subspaces of a finite Hilbert space is an orthomodular lattice*). Later, in part II of my dissertation, I will comment on the philosophical import of these two results.

LQM starts by defining two different kinds of abstract spaces, the *observation space* **τ**, and the *phase space* **Σ**. The latter is a very common notion in physics: it is an xn-dimensional space, where n is the number of individual elements that constitute a physical system **Ç**, and x is the number of degrees of freedom of every single element. The only free parameter is time. In classical mechanics the physical system **Ç** is at each instant represented by a *point* in phase space **Σ**. In quantum mechanics, as in electrodynamics, **Ç** is represented in **Σ** by a function, not a point. The quantum mechanical phase space Σ is the Hilbert space of infinite dimensions; and the quantum mechanical phase functions are the so-called wave-functions.

The other abstract space, the observation space **τ**, is defined, according to LQM, not as a mathematical space but as a *physical* space. I quote from LQM: "It follows that the most general form of prediction concerning **Ç** is that the point $(x_1, \ldots, x_n)$ determined by actually measuring $(\mu_1, \ldots, \mu_n)$ will lie on a subset **S** of $(x_1, \ldots, x_n)$-space. Hence, if we call the $(x_1, \ldots, x_n)$-spaces associated with **Ç** its 'observation-spaces' we may call the subsets of the observation-spaces associated with any physical system **Ç**, the 'experimental propositions'." (Birkhoff and von Neumann 1936: 106). So **Σ** contains mathematical entities but the observation space –let us call it **τ**–, contains actual experimental statements in physicalistic language. Van Fraassen calls these, *elementary statements*, because he wants to avoid the word *proposition*, which implies a definite truth-value and the principle of bivalence. But in the context of LQM, that is the euphemism of a logician. It is simply more appropriate to call them *propositions*, for that is precisely what they are: statements whose truth or falsity can be ascertained immediately by observation.

The convictions of LQM are not stated but they are forcefully active from the very beginning: –as quickly as possible Birkhoff and Von Neumann run into bivalent realism. Not only do they conceive of the elementary statements as propositions, they are also going to establish a connection between the mathematical phase-space and the propositions. Once they achieve that connection, the following picture is going to evolve naturally: the propositions will be the syntax of the quantum-logical language and the phase-space **Σ**, through its underlying lattice of projection operators, will provide the semantics.[12]

[12] Note already here the echoes of what will be the constant theme of section 6: As Dummett seems to have been the first to point out, the fundamental question is whether the syntax already incorporates the principle of bivalence through the notion of *proposition*. Of course, *that is precisely why* Van Fraassen does not wish to accept the description of elementary statements in terms of propositions, as it is done in LQM.

So far, we have the following scheme:

Physical System
- States ⟶ Elements of phase space. (1)
- Measurable physical magnitudes ⟶ Propositions. (2)

The aim of LQM is to construct a link between (1) and (2). The link will establish what kind of phase space corresponds to what kind of propositions. If there is something fundamentally *weird* about the algebra of projectors in phase space, then there will be something fundamentally *weird* about the logic of quantum mechanical propositions as well.

The link can be established through what Van Fraassen calls *the mapping h* and the authors of LQM call the *mathematical representative*. Let me quote the **definition of mathematical representative** given in LQM: "By the 'mathematical representative' of a subset **S** of any observation-space $\boldsymbol{\tau}$ (determined by compatible observations $\alpha_1, \dots, \alpha_n$) for a quantum-mechanical system, will be meant the set of all points f of the phase-space $\boldsymbol{\Sigma}$ of the system, which are linearly determined by proper functions f_k satisfying $\alpha_1 f_k = \Gamma_1 f_k, \dots, \alpha_n f_k = \Gamma_n f_k$,where $(\Gamma_1, \dots, \Gamma_n) \in$ **S**" (Birkhoff and von Neumann 1936: 108). This is in fact a very ingenious way to introduce the usual theory of eigenvalues and eigenfunctions of quantum mechanics: the eigenfunctions are, by definition, part of the phase-space $\boldsymbol{\Sigma}$; and notice how the meaning-content of the original propositions has been drastically reduced to comprise only eigenvalues. The passage just quoted truly constitutes a crucial step: –The meaning of the propositions has been defined by the formal apparatus of quantum mechanics. This is, of course, perfectly legitimate. But it is important to stress, as does Van Fraassen, that we are here effectively defining the semantics for the language of propositions: –we are allowing quantum mechanics to dictate the semantics for our quantum logic.

The extraordinary result of this definition is that, of course, set-theoretical inclusion relationships (and their correspondent associated logical relations) can now be applied to the propositions. Surely this is what *to provide the semantics* was meant to signify. Such is the case simply because according to the definition above the *mathematical representative* of any experimental proposition is a closed linear subspace of Hilbert space. Thus, the following statements concerning two propositions P and Q are found to be equivalent:

(1) The mathematical representative of P is a subset of the mathematical representative of Q. Symbolically: $\boldsymbol{\Omega}(P) \in \boldsymbol{\Omega}(Q)$, where $\boldsymbol{\Omega}$ stands for mathematical representative, and $\in$ is set-theoretical inclusion.

(2) P implies Q –ie. the empirical prediction of Q is certain if the empirical prediction of P is certain. Symbolically: $P \longrightarrow Q$.

Van Fraassen (1975) shows that the main achievement of LQM is to keep the language closed under the usual statement connectives, yet to hold bivalence for the elementary statements (ie. to hold propositions). In order to achieve that, the authors of LQM need to prove that negation, conjunction and disjunction, are well defined relationships in the logic of propositions. This can be done by means of the relations of set-theoretical inclusion in phase space $\mathbf{\Sigma}$ –ie. in Hilbert space: –if one can prove that negation, conjunction and disjunction are well defined for the mathematical representatives then one would have automatically obtained a proof (through the application of the mathematical representative) to the effect that negation, disjunction and conjunction are *also* well defined for the logic of propositions. So, once again, the definition of mathematical representative plays the crucial role of setting a connection between phase space $\mathbf{\Sigma}$ and observation space $\boldsymbol{\tau}$.

The mathematical representatives are subsets of Hilbert space: –The following two results are straightforward:

(3) The mathematical representative of the negative of any experimental proposition is the *orthogonal complement* $\mathbf{\Omega'}$ of the mathematical representative $\mathbf{\Omega}$ of the proposition itself.

(4) The closed linear sum $\mathbf{\Omega}_1 + \mathbf{\Omega}_2$ of any two closed linear subspaces $\mathbf{\Omega}_i$ of Hilbert space, is the orthogonal complement of the set product $\mathbf{\Omega}_1' \cdot \mathbf{\Omega}_2'$ of the orthogonal complements $\mathbf{\Omega}_i'$ of the $\mathbf{\Omega}_i$.

Therefore if one postulates, as the authors of LQM do, that (5) *the set-theoretical product of any two mathematical representative is itself a mathematical representative*, then one has defined (3): negation (complementation), (4): disjunction (linear sum) and (5): conjunction (product) for the mathematical representatives (in phase space $\mathbf{\Sigma}$) of the propositions (in observation space $\boldsymbol{\tau}$).

We have seen in the previous section that every projector is associated with a subspace of Hilbert space. Thus, every mathematical representative is correlated with a projector. Therefore, the usual quantum mechanical operations over projectors can now be used to define the operations (3), (4), (5) upon mathematical representatives in phase space $\mathbf{\Sigma}$; these latter operations will then induce logical relations upon their equally associated propositions in observation-space $\boldsymbol{\tau}$: –The lattice for projectors dictates the logic of propositions.

It is surprising that, having applied such reasoning to define the operations of negation (3), disjunction (4) and conjunction (5) upon mathematical representatives, the authors of LQM now attempt to prove their negative result, –about the failure of the distributivity law in quantum logic–, by totally different means[13]. In fact, at this stage, the argument from the projectors is straightforward[14]: Imagine **|c>** is a linear combination of **|a>** and **|b>**, so:

P [|c>] P [|a>] + P [|b>] where **+** does **not** represent sum but **linear span**.

Then: **P [|c>] ∩ P [|a>] + P [|b>] = P [|c>] ∩ P [|a>] + P [|c>] ∩ P [|b>]** is the distributive law for projectors and it obviously does not always hold, for the LHS = **P [|c>]** while the RHS = **0 + 0 = 0**. The logic of propositions follows the relations between projectors, so the first negative result of LQM is finally proved: *the lattice of quantum mechanics is not Boolean*.

At this stage, the second positive result of LQM (*The algebra of the subspaces of a finite Hilbert space is an orthocomplemented modular lattice*) can also be derived. The Ortho-complementarity part is implicit in (3): –We saw in section 3 that the projectors obey orthogonality conditions *and* given that the eigenvalues of a projector are either 1 or 0, they also obey complementarity conditions.[15] So, by (3), the logic of propositions *must* incorporate Ortho-complementarity. What about the other part of the positive result, namely modularity? Here is the proof:

Take three subspaces of Hilbert space, let us call them **a, b, c**. The condition of modularity (see section 2, definition 9) applies only when **a c**. Thus, whatever **b** is, the following holds: **a (a U b) ∩ c**. And it is always true that **b ∩ c (a U b) ∩ c**. So, in our case: **a U (b ∩ c) (a U b) ∩ c**. Any vector in **(a U b) ∩ c** can be written as **§ =** α **+ ß** (where α ∈ **a, ß** ∈ **b, §** ∈ **c**). But **ß = §** – α is in **c** (since **§** ∈ **c** and α ∈ **a** which by the initial assumption is already in **c**). Hence **§ =** α **+ ß** is in **a U (b ∩ c)**. Hence **(a U b) ∩ c a U (b ∩ c)**. The algebra of projectors is thus a modular lattice, and so must be, thus, the logic of propositions.

The authors of LQM then go on to prove that an *infinite* dimensional Hilbert space does not even obey modularity. However, I shall be content with the two results obtained so far. They constitute the basis for the philosophical elucidations about the status of logic that I shall attempt to discuss now.

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[13] They give a physical thought experiment which turns out to be based on seriously flawed arguments: we will come back to this issue in section 7.

[14] As given in Redhead (1987).

[15] For a more detailed proof, see Van Fraassen (1975: 595). In LQM the proof is taken for granted.

Part II: Philosophy Of Quantum Logic

5. Putnam's conceptual revolution restated.

The conceptual revolution can be summarized by means of Putnam's very famous analogy[16]:

$$\frac{\text{geometry}}{\text{general relativity}} = \frac{\text{quantum logic}}{\text{quantum mechanics}}$$

Quantum mechanics ought to change our logic, just as general relativity once led to a change in geometry. What were *assumed* to be necessary truths for thousands of years, in logic as in geometry, must be susceptible to revision by experiment. This extremely strong view about the revisability of logic, as put forward by Putnam, is a result of *both* the acceptance of the Birkhoff-Von Neumann interpretation *and* of a philosophical belief that Putnam presumably adopted from Quine's (1951) "*Two Dogmas of Empiricism*". What makes the strong interpretation *strong*, is the combination of bivalent realism *and* the belief that, if there is a conflict between the logic of thought and the logical structure of the world, then the former *must* change.

Is there such a conflict? Are there any classical-mechanical *truths*, –true by virtue of the canons of classical logic–, that are *false* in quantum-mechanics, –false of course according to the *same* classical canons–? If so, is there any new set of rules that would make those falsities **–***true*? Putnam swiftly suggests that the distributive law is the source of *all* such cases of logical incompatibility.

Consider *one* of those conflictive cases. The following statement **(i)** would be true in classical physics but false in quantum mechanics: *The particle A has position x and the particle A has momentum r'*. Putnam's argument is as follows: statement **(i)** results from the observation (true both in classical and quantum logic) that *the particle A has position x* and *(the particle A has momentum* r_1 or *the particle A has momentum* r_2 or … or *the particle A has momentum* r_n*)* **(ii)**. This is indeed true in the Birkhoff-Von Neumann framework (for an n-dimensional Hilbert space) if the first conjunct is true, i.e. If the particle has been found to have position **x**. By virtue of the definition (4) of disjunction, given in the previous section, the second conjunct of the expression spans the whole space **V**, and is therefore true as well.

[16] The analogy is in Putnam (1976). However, the discussion in this section borrows mostly from Putnam's (1968).

If we now assume distributivity (permitted in classical logic), we get a statement that is true in classical mechanics: (*The particle A has position x and the particle A has momentum* r_1) or (*The particle A has position x and the particle A has momentum* r_2) or … or (*The particle A has position x and the particle A has momentum* r_n). **(iii)**

Now, Putnam crucially argues, *disjunction preserves truth-values*, both in classical and quantum logic, so the truth of the above statement implies the truth of some one of the disjuncts –and that proves that our initial statement **(i)** holds true (in classical mechanics). However, it is clear from definition (5) of conjunction that **(i)** does not hold in quantum mechanics: –The intersection of the subspaces spanned by a momentum projector and a position projector is the null space (they are not commutable operators!). In the Reichenbach interpretation that would imply that the truth of **(i)** is indeterminate, but in the Birkhoff-Von Neumann interpretation –because it is founded upon bivalent propositions–, any proposition that is mapped onto the null space is a necessarily *false* statement. Why does **(i)** *not* hold true in quantum mechanics? Putnam readily gives the answer: Because quantum logic does not possess a distributive law –the derivation of statement **(iii)** from statement **(ii)** is impossible in quantum mechanics.

A number of assumptions have been presupposed so far: first, the Birkhoff-Von Neumann picture has been taken for granted; second, it has been assumed that the distributive law fails but conjunction, for example, still applies. In the Putnam version of LQM**,** *all* the usual logical connectives and rules apply, **–***except for* distributivity–, and the apply in exactly the same usual way.

So far Putnam has just explored the consequences of LQM in a physicalistic language. Putnam's second belief (the belief in the fallibility of *all* knowledge, included logic) comes in at this stage in the form of a dilemma: "Two propositions that are equivalent according to classical logic [say, **(i)** and **(ii)**] are mapped onto different subspaces of Hilbert space… **Conclusion**: the mapping is nonsense, *or, we must change our logic*." (1968: 179) [my italics]. Putnam's mapping is our old friend, the mathematical representative. Nobody with a positive attitude towards science would deny the legitimate aspiration to build a connection between the mathematical structure of a theory and its empirical statements. So, Putnam, confidently, takes it in the latter option: *we must change our logic.*

It is remarkable that, at this stage in Putnam's discussion, there is already a visible step in the argument that may be cast into doubt. Moreover, it is the very first step: –the initial conjunction of beliefs (first, the belief in the correctness of the Birkhoff-Von Neumann LQM interpretation of quantum logic; second, the belief in the revisability of logic, –given that there is a conflict between logic and the "world"). Of course, none of these beliefs are *necessary***.** First, it may be argued that the LQM interpretation is not to be taken for granted (Popper and Reichenbach seem to have manifestly rejected it). Even if the LQM interpretation (including the use of propositions and the mathematical representative), is accepted for quantum logic, it is still not clear why the other belief, *–logic is revisable by empirical means–*, has to be similarly accepted. Even if both beliefs may be somehow justified by different arguments, independently of each other, (LQM, say, because it goes along with usual practice in mathematical physics; *empirical logic*, say, because it is a very sound and healthy philosophical wisdom)–, that does not imply that they are consistent with each other.

This is the central question in the Putnam-Dummett debate. Putnam is next going to attempt to show that LQM somehow implies that *logic is empirical*. Dummett will then argue that, in fact, LQM does not only **not** imply an empirical conception of logic, but, on the contrary, in a certain way, LQM is actually incompatible with an empirical view of logic. First, in the remaining part of this section, I review Putnam's arguments in this direction; then, in the following section, I analyze Dummett's answer, and I attempt to focus the full debate in the terms of the 3-layer epistemological model that I set up in section 1.

Putnam is very much aware of the fact that, in order to prove logic empirical, he must show that, in going from classical to quantum logic, the usual logical connectives (**¬, v, &**) have not just simply changed their meaning. What he has in mind can be put very crudely by means of the following example: Suppose one has a theory that contains just the following two axioms:

1. **a** and **(a** or **b)** ⟶ **(a** or **b).**
2. **(a** and **a)** ⟶ **a.**

where **or, and** stand for the usual classical **v¸ &**. A law in this theory would be the following:

3. **a** ⟶ **a** or **b.**

If this law is found to be false by empirical means, then a conceptual revolution as occurred: a logical law has been proved false! But, –as Putnam is well aware of–, a sceptic might argue that there is no such revolution: actually, no law has been found wrong –it is the connectives **or, and** that have *changed their meaning.* This can be envisaged in our very crude example as follows: If one makes **or** stand not for the classical **v**, but for a new logical connective, –let us call it the **equalizer, w**–, defined by the following truth-table,

a	T	T	F	F
b	T	F	T	F
a w b	T	F	F	T

then, it is clear that, under such change of meaning of the connective **or**, the law 3.

a ⟶ **a** w **b** fails to hold, while both axioms remain true.

Putnam thinks that the change-of-meaning argument, when generalized to quantum logic, is used by philosophers in order to *minimize* the revolution of quantum logic. However, he now finds himself in an awkward position: The change-of-meaning argument does not need a definition of what the new quantum logical connectives stand for; in order to *minimize* the revolution, it only has to be shown that whatever these connectives mean, *they do not mean what they used to mean*. Putnam now wants to prove the opposite – that *they mean the same*. Following Finkelstein, he tries to give an operational definition for the logical connectives: given that the operations which define the meanings of the connectives do not change, when going from classical to quantum mechanics, the he can claim that the meanings themselves cannot possibly have changed.

In LQM, negation (3), disjunction (4) and conjunction (5) were extracted from the lattice. In order to be so, implication (2) had to be previously defined. Putnam seems to think that if he can show that implication (2) may have an operational meaning, then that would automatically set the operational meaning for *all* (3), (4) and (5) –through the lattice definitions of orthocomplement, meet and join, respectively. His argument is based on the notion of *test*: "Let us pretend that to every physical property P there corresponds a test T such that something has P just in case it 'passes' T (i.e. it *would* pass T, if T were performed) […] We define the following natural 'inclusion' relation among tests:

$T_1 \in T_2$ just in case everything that 'passes' T_1 'passes' T_2

This inclusion relation may be operationally tested […] Take out a fair sample S_1, an apply test T_1 to every member of the sample […] take a different fair sample S_2 and apply T_2. If all the elements of S_2 pass T_2, the hypothesis that "all the things that pass T_1 also pass T_2" has been confirmed." (Birkhoff and von Neumann 1936: 195)

And, if the operational meaning of the logical connectives **v, &, ¬**, remains the same, in quantum as in classical logic, then logic would have been proved to be empirical.

6. Meaning, realism and the minimized revolution

Of course, opponents to the strong interpretation of quantum logic still have the same powerful counter-argument available: –they can try to show that the logical connectives, however defined, *do* actually, in the transition from the old logic to the new logic, undergo a change of meaning. Furthermore, it may be sufficient to prove that the way in which these connectives are derived from the set-theoretic inclusion relation changes, when one goes from a Boolean algebra to an orthomodular lattice (because even if implication may be derived operationally from set-theoretical inclusion, it has only been *assumed* that, according to definitions (3), (4) and (5), negation, disjunction and conjunction are always defined, or defined *in the same way*, in terms of set-theoretical inclusion). And, in fact, it is enough with a proof to the effect that any *one* connective has undergone such change, in its derivation from the basic algebraic set structures.

An argument that shows that a change *does* actually take place, –when shifting from Boolean lattices to orthomodular lattices–, in the meaning of the logical negation (¬), has been presented by Bell and Hallett (1982). Their reasoning is neat and can be put in a nutshell as follows: Given a certain condition for the definition of *invariance of meaning* of a term, this definition can be seen to apply to the terms expressed by **v** and **&** as they stand in the orthomodular lattice of quantum logic; however, the same definition does not apply to ¬.

Bell and Hallett express their condition for the invariance of meaning as follows: "If two terms **t** and **t'** are defined in terms of the primitives a, b, … etc. in non-equivalent ways, or if one is so definable and the other is not, then they have different meaning relative to a, b…" "More particularly" –they continue–, "if two structures L and L' both have the primitives a, b, … etc. and **t** is definable in terms of a, b, … in one and not in the other then we will assume that **t** has shifted its meaning in the passage from one to the other" (Bell and Hallett 1982: 363). In an orthomodular lattice, the operations **v, &** can be defined in terms of set-theoretical inclusion (≤), *just* in the same way as they can be defined in a Boolean algebra, ie. by the equivalence with **meet** and **join**:

$$\mathbf{a \,\&\, b = c \leftrightarrow \{x: x \leq c\} = \{x: x \leq a\} \cap \{x: x \leq b\}}.$$
$$\mathbf{a \vee b = c \leftrightarrow \{x: c \leq x\} = \{x: a \leq x\} \cap \{x: b \leq x\}}.$$

In a Boolean algebra, negation (¬) can, of course, be also defined in terms of ≤ and of **&** (and hence of ≤), by the equivalence with complementation, *:

$$\mathbf{a = b^* \leftrightarrow \{x: x \leq a\} = \{x: b \,\&\, x = 0\}} \qquad \mathbf{(*)}$$

However, in a non-Boolean Ortho-lattice, –whether modular or not–, ¬ *cannot* be defined in terms of **≤,** as in (*). The reason is the following: while a Boolean algebra is complemented, an Ortho-lattice is always *ortho*complemented. In an Ortho-lattice, negation (¬), can only be defined in terms of some *orthogonality* relation, ⊥. The orthocomplement is given by:

$$\mathbf{a = b_{*}\ \{x: x \leq a\} = \{x: x \perp b\}} \qquad \mathbf{(**)}$$

And, of course, in quantum logic, negation (¬), is defined as the orthocomplement. Thus, it is clear that the classical logical negation (*) and the quantum logical negation (**) are by no means the same, nor is the term ¬ defined upon the same primitives (**≤, &** or ⊥) in both of them.

If there is variance of meaning, then, what does quantum logic have to say to the philosophy of logic? Or is quantum logic just a syntactical game –fixing the set of rules that will give consistency to some algebraic structures? Dummett (1976) has provided a further insight by showing that the attempt to change the logic of thought is based on a philosophical longing for epistemological realism. He has emphasized that a fundamental aspect of Putnam's program is the belief that, if logic turned out to be revisable, then quantum logic would presumably establish a realistic interpretation of quantum mechanics. This belief of Putnam's can be schematically represented as follows:

Classical universe:	**Pc + Mc + Lc**
Quantum universe:	**Pq + Mq + Lc**
Alternative quantum universe:	**Pq + Mc' + Lq**

The classical description contains the classical physics (**Pc**), the classical realistic metaphysics (**Mc**), and the classical logic (**Lc**). The current quantum description offered by, say, the Copenhagen interpretation, contains the quantum physics (**Pq**), the quantum, complementary (non-realistic) metaphysics (**Mq**), and the old logic (**Lc**). The new quantum framework, advocated by Putnam, would *rescue* some kind of realistic metaphysics (**Mc'**), by paying the price of having to change the logic of thought (from **Lc** to **Lq**).

So, –Dummett claims–, it is the search for a realistic metaphysical interpretation of theoretical physics, which prompts the proposal to revise our logic. But, argues Dummett, the result of Putnam's proposal turns out to be *precisely* the opposite to the intended one. A revision of logic, as implied by the strong interpretation of quantum logic, would effectively *kill* logical realism, –i.e. it would deny the possibility of associating every physical statement to a bivalent algebra of truth-values. This is the case because, in quantum logic, the connectives change their meanings. In fact, if the distributive law is dropped, all the classical logical connectives necessarily will vary their meanings in one particular respect: their truth-values will not be capable of reconstruction upon the law of excluded middle. Dummett argues as follows: if the law of distributivity does not hold, then truths does *not* distribute upon conjunction or disjunction. It is not a possibility to drop a law in logic, and still pretend to keep the same semantical relations –ie. to keep the same definition of *truth*.

For example, Putnam would argue that the truth of statement **(ii)** of the previous section, implies the truth of one of the momentum disjuncts, ie. there is a momentum that the particle **A** *has*. This is how the realistic metaphysics (**Mc'**) comes in: particles are supposed to possess definite momenta, even if their position has been previously determined. However, one *cannot* know what this momentum is, –say by affirming statement **(iii)**–, for such knowledge would require the use of the distributive law, and that would result in a quantum-logical contradiction. But –Dummett argues–, Putnam is allowing himself, at this point, to use the classical definition of truth, –namely, he is distributing truth upon disjunction–, in order to prove that the particle *has* a momentum, –even if he immediately concedes that this momentum is impossible to know. Now, –argues Dummett–, if we reject the law of distributivity then we must dispense altogether with it from the semantics: we cannot pretend to keep on distributing *truth* upon disjunction. It is not legitimate to take the law out from syntax of logical inference and, yet, to keep on using it for the semantics of truth. It is therefore Putnam himself, and not his opponents, who *minimizes* the revolution of quantum logic: "It is Putnam who cannot for a very long time, appreciate the logic employed in quantum mechanics; it is Putnam […] who 'smuggles in' distributivity." (Dummett 1976: 273).

So, there is no possible return to a realistic bivalent metaphysics. The framework that accommodates quantum logic looks more like: **Pq + Mq + Lq**, –there doesn't seem to be much improvement over the Copenhagen interpretation. Or does it? My opinion is that there is improvement: first, quantum logic helps to understand how extraordinary quantum theory is; and, second, it helps to understand how powerful the propositional calculus is. In this sense, quantum logic ought to be given a heuristic role, but not a normative one; –this takes me back to the 3-layer epistemological model.

It is remarkable that Putnam's argument effectively annihilates the distinction that I drew, in section 1, between different forms of realism. If the connection between the phase space **Σ** and the observation space **τ** is established *and* the elementary statements are propositions, then *either* the truth-values of the propositions are dictated by the lattice that corresponds to **Σ**, or the truth-values of the propositions dictate what the lattice is like. And, if the truth-values of the propositions are determined *a priori* and *independently* by experiment, then an empiricist conception of logic ought to be accepted: the lattice is determined by experiment. In the epistemological model of section 1, this is equivalent to establishing a connection between layer 3 (empirical data) and layer 1 (thought), such that the empirical data determines the logic of thought, ie. logic depends upon the world (the opposite does not necessarily hold). Of course, that is the case only if the elementary statements are propositions. And *this* is the problematic assumption; for to accept propositions is to accept metaphysical bivalence, and bivalence is not the right metalogical principle to apply to orthomodular lattices.

How do we get out? –or, better, how *did* we get in? If we disallow a priori the possibility of a connection between layers (1) and (3), then we are fine. That's what the 3-layer model does for you –it establishes that the mathematical representative is not what it looks like according to Putnam's programme: it is not a connection between the world and the logic of thought. It is just a connection between the empirical data (e-values in **τ** space) and the theory (e-functions in **Σ** space). No inferences are to be made from layer (3) to layer (1). There is never a straight path from the empirical data (3) to our thinking (1). Sooner or later, we have to pass by layer (2), which contains our theories. This is, also, the sense in which Dummett's (1976) paper is a Wittgensteinian piece of work, for it was Wittgenstein in the Tractatus, who argued that there is a fundamental cleavage between logic (the world of "facts"), and the world of material objects.

7. The revolution maximized

The following awkward situation has emerged: LQM presents a quantum logic, the logic induced by an orthomodular lattice, –the lattice of Hilbert space. This is a logic founded upon the principle of bivalence: propositions are either true or false according to whether they are mapped onto the whole space or the null space. However, a result of such logic is the breakdown of the distributive law. The distributive law is necessarily implied by bivalence. Thus, if this LQM logic, –as it turns out, without distributive law–, is implemented as the logic of thought, then, the foundations, upon which the results of LQM were originally established, are found to be misplaced.

Popper (1968), in an otherwise severely criticized paper, seems to have had the intuition that there was to be trouble, in LQM, with the principle of bivalence.

The history of the argument is indeed bizarre and worth going through: Poper set himself the task of scrutinizing LQM, particularly the thought experiment designed by the authors to prove the failure of the distributive law in quantum logic. In doing so, Popper found out that the authors of LQM *implicitly* rejected the same law of excluded middle that they *explicitly* adopted for their propositions.

Now, the thought experiment turns out to contain a serious flaw, as Popper himself was able to prove. The experiment goes as follows:

> "The distributive law may break down in quantum mechanics […] If **a** denotes the experimental observation of a wave-packet **φ** on one side of a plane in ordinary space, **a'** correspondingly the observation of **φ** on the other side, and **b** the observation of φ in a state symmetric about the plane, then (as one can readily check): $\mathbf{b \cap (a \cup a') = b \cap 1 = b}$ $\mathbf{b \cap a = b \cap a' = (b \cap a) \cup (b \cap a')}$" (Birkhoff and von Neumann 1936: 113).

But – as Popper (1968) quickly pointed out–, "**a'** (the complement of which according to Birkhoff and Von Neumann is the ordinary classical set-theoretic orthocomplement) does not 'correspondingly' denote its observation of position on the other side. **a'** denotes, rather, the property 'not on the one side'" and "this is perfectly compatible with the property denoted earlier by **b**", so that "the thought experiment by Birkhoff and Von Neumann breaks down" (Popper 1968: 685).

In fact, the thought experiment seems totally unnecessary and inappropriate, in the context of LQM. The proof of the failure of the distributive law may be easily derived from the algebra of projectors, as shown on pages 22-23 in this dissertation. There is no need whatsoever for a thought experiment. So, what is it doing, in such an otherwise brilliant and consistent paper as LQM? Popper suggests that the only way to make sense of the thought-experiment is by assuming the rejection of the law of the excluded middle (a direct consequence of the rejection of metaphysical bivalence) by Birkhoff-Von Neumann, "for the **b** of their thought experiment is clearly intended as a third possibility, incompatible with either **a'** or **a**".

Now, this implicit rejection of bivalence is somewhat surprising, for such is precisely the result of Dummett's analysis of Putnam's paper. The authors of LQM reject bivalence, –which they had been blindly advocating all the way– *precisely* when they set to prove their negative result. How are we going to interpret this fact? Are we going to assume, as does Popper, that this was "no more than a simple slip –one of those slips which, once in a lifetime, may happen even to the greatest mathematician"? May it be suggested that, on the contrary, there might be some particular reason for this thought experiment? It is a conceivable possibility that the authors of LQM somehow *had an intuition* about what direction quantum logic was about to take –not just the simple rejection of a law, but a real revolution, namely, a total rejection of metaphysical bivalence.

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